Journal of Applied Mathematics and Mechanics 72 (2008) 461–467

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The deformation of a half-space with a gradient elastic coating under arbitrary axisymmetric loading☆

S.M. Aizikovich*, L.I. Krenev, I.S. Trubchik

Rostov-on-Don, Russia

ARTICLE INFO

Article history:
Received 22 May 2007

ABSTRACT

An analytical-numerical method of solving the Neumann boundary-value problem for an elastic half-space with a gradient elastic coating is proposed. The problem is formulated and the construction of the fundamental solution (Green's function) is described. The method enables a solution of the problem to be obtained for a fairly wide class of types of non-uniformity of the medium, and effects related to the non-uniformity are investigated analytically. A procedure for calculating the displacement, stress and strain fields is described. Particular attention is devoted to analysing the mechanical characteristics in the transition region from the coating to the elastic substrate.

Non-uniform coatings with elastic properties that vary smoothly with depth, so-called gradient (or functionally gradient) coatings, enable one to avoid stress concentrations in the region where the coating adjoins the elastic substrate, and hence the use of such coatings is one of the promising areas in tribology. However, mathematical models of multilayered materials developed previously do not describe functionally gradient coatings sufficiently accurately,[1–3] and both quantitative and qualitative differences are observed in the behaviour of functionally gradient materials compared with uniform or multilayered materials.

Among the non-uniform coatings with elastic properties that vary smoothly with depth, functionally gradient coatings, in which the gradient of the change in the elastic properties changes sign along one of the coordinates, are of particular interest. An example are coatings, the stiffness of which is a maximum on the surface and decreases with depth to a certain value, and then, inversely, increases as one approaches the substrate (coatings with a non-monotonic change of the elastic properties with depth). In view of the complexity of the mathematical modelling of media whose properties vary continuously along at least one coordinate, in the majority of existing publications in most cases it is multilayered materials (or piecewise-uniform materials) that have been investigated.

The solution of contact problems for a functionally gradient half-space with an arbitrary variation of the non-uniformity with depth was constructed in Refs 4–9 using a bilateral asymptotic method. It was proved in Ref. 10 that the approximate analytical solution constructed is an asymptotically accurate solution of the initial equation both for small and large values of the characteristic geometrical parameter of the problem.

Our attention will be concentrated on analysing the fields of elastic displacements and stresses, in particular, on investigating the effect of a change in the elastic properties of the material on the internal stresses in the surface layers of a material with a multilayered or functionally gradient coating when indented. The methods employed give stable numerical results for different forms of multilayered and functionally gradient materials.

The stresses inside the material have a different form depending on the extent to which the elastic properties of the coating differ from those of the base or from the gradient of the elastic properties in a functionally gradient material. For example, it was shown in Refs 11 and 12 that, for certain materials and operating conditions, the highest tensile stresses occur in the region where the coating and the substrate are in contact, rather than on the surface of the coating.

In this paper we analyse the mechanical characteristics of functionally gradient materials in the transition region from the coating to the elastic substrate, including for contact-interaction problems.

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 4, pp. 644–651, 2008.

* Corresponding author.
E-mail address: trubchik@math.rsu.ru (S.M. Aizikovich).

0021-8928/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2008.08.007

1. Formulation of the boundary-value problem for a half-space, continuously non-uniform in depth for forces specified on its surface

We will consider an elastic half-space, the elastic characteristics of which vary continuously with depth in a layer of thickness H adjacent to the surface, and are then stabilized and remain constant. We connect a cylindrical system of coordinates (r, φ, z) with the half-space. We will assume that the forces adjacent to the half-space are produced either by the action of a punch, having the shape of a convex solid of revolution, or by a force $p(r)$, distributed inside a circle of radius a, and hence are symmetrical about the coordinate axes (Fig. 1). Hence, the displacements, strains and stresses are independent of the angular coordinate φ. We will also assume that the Lamé coefficients $M = M(z)$ and $\Lambda = \Lambda(z)$, connected with Young's modulus E and Poisson's ratio ν by the relations

$$\mathrm{M} = \frac{E}{2(1+\nu)}, \quad \Lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}; \quad E = \frac{\mathrm{M}(2\mathrm{M}+3\Lambda)}{\mathrm{M}+\Lambda}, \quad \nu = \frac{\Lambda}{2(\mathrm{M}+\Lambda)}$$

vary with depth in the surface layer of thickness H, and remain constant when $z \le -H$

$$\begin{aligned}
&\mathrm{M}(z) = \mathrm{M}(-H) = \mathrm{M}^s, \quad \Lambda(z) = \Lambda(-H) = \Lambda^s, \quad -\infty \le z \le -H \\
&\mathrm{M}(z) = \mathrm{M}^c(z), \quad \Lambda(z) = \Lambda^c(z), \quad -H \le z \le 0 \\
&\mathrm{M}^c(-H) = \mathrm{M}^s, \quad \Lambda^c(-H) = \Lambda^s \\
&\min\Lambda(z) \ge \Lambda^* > 0, \quad \max\Lambda(z) \le \Lambda^* < \infty, \quad \min\mathrm{M}(z) \ge \mathrm{M}^* > 0, \quad \max\mathrm{M}(z) \le \mathrm{M}^* < \infty
\end{aligned} \tag{1.1}$$

The minimum and maximum are taken when $z \in (-\infty, 0)$. The superscript s corresponds to the underlying uniform half-space, while the superscript c corresponds to the non-uniform layer, and Λ^*, Λ^*, M_* and M^* are constants.

In this case the equilibrium equations have the form

$$\frac{\partial}{\partial r}(r\sigma_r) + r\frac{\partial\tau_{rz}}{\partial z} - \sigma_\varphi = 0, \quad \frac{\partial}{\partial r}(r\tau_{rz}) + r\frac{\partial\sigma_z}{\partial z} = 0, \quad \frac{\partial}{\partial r}(r^2\tau_{rz}) + r^2\frac{\partial\tau_{\varphi z}}{\partial z} = 0 \tag{1.2}$$

The first two equations of system (1.2) describe the axisymmetric stress state, which occurs, for example, due to the action of a load normal to the surface, while the last equation is the equilibrium equation of the half-space, twisted by the shear force.

Using Hooke's law, we can write the relation between the elastic stresses and the displacements in the form

$$\begin{aligned}
&\sigma_r = 2\mathrm{M}\frac{\partial u}{\partial r} + \Lambda\theta, \quad \sigma_\varphi = 2\mathrm{M}\frac{u}{r} + \Lambda\theta, \quad \sigma_z = 2\mathrm{M}(z)\frac{\partial w}{\partial z} + \Lambda\theta, \quad \theta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \\
&\tau_{rz} = \mathrm{M}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right), \quad \tau_{r\varphi} = \mathrm{M}\left(\frac{\partial \upsilon}{\partial r} - \frac{\upsilon}{r}\right), \quad \tau_{\varphi z} = \mathrm{M}\frac{\partial \upsilon}{\partial z}
\end{aligned} \tag{1.3}$$

Using the formulae relating the components of the strain and stress tensors in the axisymmetric problem, we obtain the following system of Lamé equations

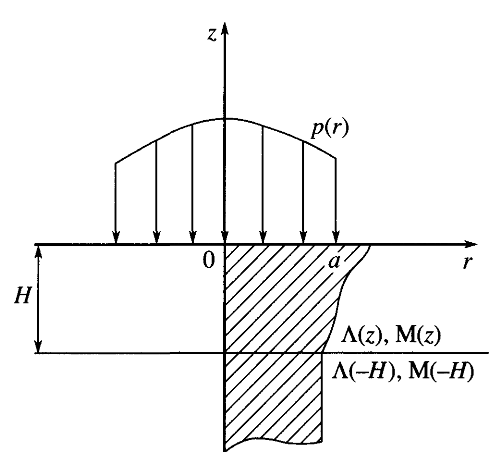

Fig. 1.

$$M(z)\left(\nabla^2 u - \frac{u}{r^2}\right) + (M(z) + \Lambda(z))\frac{\partial\theta}{\partial r} + M'(z)\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right) = 0$$

$$M(z)\nabla^2 w + (M(z) + \Lambda(z))\frac{\partial\theta}{\partial z} + 2M'(z)\frac{\partial w}{\partial z} + \Lambda'(z)\theta = 0$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}, \quad M'(z) = \frac{dM(z)}{dz}, \quad \Lambda'(z) = \frac{d\Lambda(z)}{dz} \tag{1.4}$$

As described above, a surface load is applied in a limited region, namely inside a circle of radius a. We will consider the case of an arbitrary load, normal to the surface $z=0$

$$z = 0:\ \sigma_z(r,0) = \begin{cases}-p(r), & 0 \le r \le a,\\ 0, & a < r < \infty,\end{cases} \quad \tau_{rz}(r,0) = 0, \quad 0 \le r < \infty \tag{1.5}$$

At the boundary of the non-uniform layer with the uniform half-space, when $z=-H$, the following displacement and stress matching conditions must be satisfied

$$z = -H:\ \begin{aligned}&\sigma_z^c(r,-H) = \sigma_z^s(r,-H), \quad \tau_{rz}^c(r,-H) = \tau_{rz}^c(r,-H)\\ &u^c(r,-H) = u^s(r,-H), \quad w^c(r,-H) = w^s(r,-H)\end{aligned} \tag{1.6}$$

When $(r, -z) \to \infty$ the displacements, strains and stresses disappear.

Hence, we have formulated the following Neumann problem for a non-uniform half-space: it is required to find the displacements, strains and stresses in a half-space, satisfying Eqs (1.4), the specified distribution of the forces at the boundary (1.5) and (1.6) and the condition of attenuation at infinity.

2. Construction of the fundamental solution for a half-space, non-uniform in depth

We will find a solution for the displacements u and w in the form of Hankel integrals

$$u(r,z) = -\int_0^\infty U(\gamma,z)J_1(\gamma r)\gamma d\gamma, \quad w(r,z) = \int_0^\infty W(\gamma,z)J_0(\gamma r)\gamma d\gamma \tag{2.1}$$

We substitute these expressions into system (1.4) and equate the integrands to zero. We obtain a system of ordinary differential equations, which we can write in matrix form using the vector representation for the transformant

$$\mathbf{x}^T = (x_1, x_2, x_3, x_4), \quad x_1 = U, \quad x_2 = U', \quad x_3 = W, \quad x_4 = W'$$

clearly separating the parts corresponding to the non-uniform surface layer and the uniform base:

$$\frac{d\mathbf{x}^c}{dz} = \mathbf{A}^c\mathbf{x}^c, \quad -H \le z \le 0 \tag{2.2}$$

$$\mathbf{A}^c = \left\| \begin{matrix} 0 & 1 & 0 & 0\\ \gamma^2\frac{\kappa+1}{\kappa-1} & -\eta & -\gamma\eta & -\gamma\frac{2}{\kappa-1}\\ 0 & 0 & 0 & 1\\ \gamma\eta\frac{3-\kappa}{\kappa+1} & \gamma\frac{2}{\kappa+1} & \gamma^2\frac{\kappa-1}{\kappa+1} & -\eta \end{matrix} \right\|$$

$$\frac{d\mathbf{x}^s}{dz} = \mathbf{A}^s\mathbf{x}^s, \quad -\infty < z \le -H$$

$$\mathbf{A}^s = \left\| \begin{matrix} 0 & 1 & 0 & 0\\ \gamma^2\frac{\kappa+1}{\kappa-1} & 0 & 0 & -\gamma\frac{2}{\kappa-1}\\ 0 & 0 & 0 & 1\\ 0 & \gamma\frac{2}{\kappa+1} & \gamma^2\frac{\kappa-1}{\kappa+1} & 0 \end{matrix} \right\|$$

$$\kappa = 1 + 2M/(M+\Lambda), \quad \eta = M'/M = E'/E \tag{2.3}$$

Boundary conditions (1.5) and (1.6) take the form

$$x_4^c(\gamma,0)-\gamma\frac{(3-\kappa)}{(\kappa+1)}x_1^c(\gamma,0) = -\frac{(7-\kappa)(\kappa-1)}{2E(0)(\kappa+1)}P(\gamma)$$

$$\gamma x_3^c(\gamma,0)+x_2^c(\gamma,0) = 0,\; x_k^c(\gamma,0) = x_k^s(\gamma,-H),\; k=1,\ldots,4;\; P(\gamma) = \int_0^a p(\rho)J_0(\gamma\rho)\rho d\rho \tag{2.4}$$

We will write the general solution of system (2.3) for a uniform half-space

$$z\le -H,\quad \mathrm{M}' = \Lambda' = 0,\quad \mathrm{M}>0,\quad \Lambda>0$$

as follows:

$$x_1^s(\gamma,z) = (d_1+\gamma z d_2)e^{\gamma z},\quad x_2^s(\gamma,z) = (d_1+(1+\gamma z)d_2)\gamma e^{\gamma z}$$

$$x_3^s(\gamma,z) = (d_1+(-\kappa+\gamma z)d_2)e^{\gamma z},\quad x_4^s(\gamma,z) = (d_1+(1-\kappa+\gamma z)d_2)\gamma e^{\gamma z} \tag{2.5}$$

where d_1 and d_2 are arbitrary functions of the parameter γ.

Using the method of modulating functions,[13] we seek a solution of system (2.2) in the form

$$\mathbf{x}^c(\gamma,z) = d_1(\gamma)\mathbf{a}_1(\gamma,z)e^{\gamma z}+d_2(\gamma)\mathbf{a}_2(\gamma,z)e^{\gamma z} \tag{2.6}$$

The vectors $a_1(\gamma,z)$, $a_2(\gamma,z)$ are found from the solution of the Cauchy problems

$$\frac{d\mathbf{a}_i}{dz} = \mathbf{A}^c\mathbf{a}_i-\gamma\mathbf{a}_i,\; -H\le z\le 0,\quad i = 1,2$$

$$z = -H\text{: } \mathbf{a}_1(\gamma,-H) = (1,\gamma,1,\gamma),\quad \mathbf{a}_2(\gamma,-H) = (-\gamma H,\gamma-\gamma^2H,-\kappa-\gamma H,\gamma-\kappa\gamma-\gamma^2H) \tag{2.7}$$

The constants d_1 and d_2 are found from conditions (2.4), where $a_i^k(\gamma,z)(k=1,2,3,4)$ is the k-th component of the vector $a_i(\gamma,z)$ (i=1,2). We have

$$d_1(\gamma) = P^*(\gamma)M_2(\gamma)\Delta^{-1},\quad d_2(\gamma) = -P^*(\gamma)M_1(\gamma)\Delta^{-1}$$

$$\Delta = M_2(\gamma)N_1(\gamma)-M_1(\gamma)N_2(\gamma),\quad P^*(\gamma) = -P(\gamma)/(2\mathrm{M}(0)+\Lambda(0))$$

$$N_k(\gamma) = a_k^4(\gamma,0)-\frac{3-\kappa}{\kappa+1}\gamma a_k^1(\gamma,0),\quad M_k(\gamma) = a_k^2(\gamma,0)+\gamma a_k^3(\gamma,0)$$

We finally obtain the following expressions for the components of the vector $x^c(\gamma,z)$ when $H\le z\le 0$

$$x_k(\gamma,z) = P^*(\gamma)\Delta^{-1}\Delta_k(\gamma,z)e^{\gamma z},\quad k = 1,\ldots,4$$

$$\Delta_k(\gamma,z) = M_2(\gamma)a_1^k(\gamma,z)-M_1(\gamma)a_2^k(\gamma,z) \tag{2.8}$$

We introduce the following notation

$$I_{km} = I_{km}(r,z) = \int_0^\infty x_k(\gamma,z)J_m(\gamma r)\gamma d\gamma,\quad I_{km}^1 = I_{km}^1(r,z) = \int_0^\infty x_k(\gamma,z)J_m(\gamma r)\gamma^2 d\gamma$$

Then, using formulae (1.3) and (2.1), the distribution of the displacements, strains and stresses for a half-space, non-uniform in depth, when an arbitrary asymmetric load acts on its surface, can be represented in the form

$$u(r,z) = -I_{11},\quad w(r,z) = I_{30}$$

$$\varepsilon_r(r,z) = -I_{10}^1+\frac{1}{r}I_{11}(r,z),\quad \varepsilon_z(r,z) = I_{40},\quad \varepsilon_\varphi(r,z) = -\frac{1}{r}I_{10},\quad \varepsilon_{rz} = -\frac{1}{2}(I_{21}+I_{31}^1)$$

$$\sigma_r(r,z) = \frac{2\mathrm{M}}{1-2\nu}\left(\nu I_{40}-(1-\nu)I_{10}^1+\frac{1}{r}I_{11}\right)$$

$$\sigma_z(r,z) = \frac{2\mathrm{M}}{1-2\nu}\left((1+2\nu)\frac{1}{r}I_{11}+\nu(I_{40}(r,z)-I_{10}^1)\right)$$

$$\sigma_\varphi(r,z) = \frac{2\mathrm{M}}{1-2\nu}((1-\nu)I_{40}-\nu I_{10}^1),\quad \tau_{rz}(r,z) = -\mathrm{M}(I_{21}+I_{31}^1) \tag{2.9}$$

3. Numerical analysis of the solution for some characteristic distributions of Young's modulus

The values of the components of the displacement, strain and stress fields are constructed using numerical integration from formulae (2.9) at specified points (r, z) of a half-space non-uniform in depth. The infinite region of integration with respect to γ is split into sections in which the integrand has constant sign. The functions $L_k(\gamma, z)$ can change sign for γ and z close to zero, and then tend monotonically to zero, so that the roots of the integrand in relations (2.9) are identical with the roots of Bessel functions, which are easily determined. The calculation is carried out until the modulus of the increment of the function exceeds a specified (fairly small) value. Hence, we can integrate by withdrawing a certain distance from the boundary of the half-space. On the boundary itself the solution is sought by the method described earlier in Ref. 4.

We will consider numerical examples for some characteristic forms of non-uniformity of the base. Poisson's ratio will be assumed to be constant: $\nu = 1/3$, and Young's modulus in the non-uniform coating will be assumed to vary with depth as given by the following relations (Fig. 2)

$$E(\bar{z}) = \begin{cases} E_0 f_n(\bar{z}), & -1 \le \bar{z} \le 0 \\ E_0, & \bar{z} < -1 \end{cases}, \quad n = 1, \ldots, 6; \quad \bar{z} = \frac{z}{H}$$

$$f_1(\bar{z}) = \frac{7}{2}, \quad f_2(\bar{z}) = \frac{2}{7}, \quad f_3(\bar{z}) = \frac{7}{2} + \frac{5}{2}\bar{z}, \quad f_4(\bar{z}) = \frac{2}{7} + \frac{5}{7}\bar{z}, \quad f_5(\bar{z}) = 1 - \frac{5}{2}\sin\pi\bar{z},$$

$$f_6(\bar{z}) = \frac{7}{2} + \frac{5}{2}\sin\pi\bar{z} \tag{3.1}$$

An analysis of the numerical functions $\gamma x_k(\gamma, \bar{z})$ – the transformant of the kernel of integral equations (2.8) – shows that $\gamma x_k(\gamma, \bar{z}) \neq 0$ for $\gamma > 100$, and the procedure for the numerical construction of $\gamma x_k(\gamma, \bar{z})$ when $\gamma > 100$ must be continued up to the necessary level of accuracy. Moreover, the functions $\gamma x_1(\gamma, \bar{z})$ and $\gamma x_2(\gamma, \bar{z})$ change sign when γ and $\bar{z}$ change. It is well known that a Bessel function decreases as $\gamma^{-1/2}$ as $\gamma \to \infty$, and hence for small $\bar{z}$ and $\bar{r} = r/a$ and large λ it is difficult to make an accurate calculation of the integrals in expressions (2.8) for the functions $\gamma x_1(\gamma, \bar{z})$ and $\gamma x_2(\gamma, \bar{z})$.

It was mentioned in Ref. 14 that A. N. Dinnik, when developing the Hertz problem, analysed the distribution of elastic deformations of the sample along the direction of indentation of an axisymmetric punch. He summed the elementary deformations deeply from the surface of the sample and found discrete values of the sums for fixed thicknesses H of the surface layer, normalized to the radius of the contact area a. The ratio of this total deformation w_H to the experimentally recorded value of w, which summed the deformation over the whole depth of the half-space, turned out to be equal to 0.937 at a depth of $10a$ and 0.970 at a depth of $20a$.

In Fig. 3 we show graphs of w_H/w for $n = 1, 2, 3, 4$, for the forms of non-uniformity (3.1). The dashed curve corresponds to the case of a uniform medium.

In Fig. 4 we show the distributions of the intensity of the deviator stresses

$$\sigma_e = \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}/\sqrt{2}$$

$$\sigma_2 = \sigma_\varphi(\bar{r}, \bar{z}), \quad \sigma_{1,3} = [\sigma_r(\bar{r}, \bar{z}) + \sigma_z(\bar{r}, \bar{z}) \pm \sqrt{(\sigma_r(\bar{r}, \bar{z}) + \sigma_z(\bar{r}, \bar{z}))^2 + 4\tau_{rz}^2(\bar{r}, \bar{z})}]/2$$

It can be seen that the case $n = 4$ represents a softer coating, with a smoothly increasing value of Young's modulus with depth. The case $n = 5$ is also interesting; for this value there is no stress concentration in the transition zone from the non-uniform surface layer to the uniform

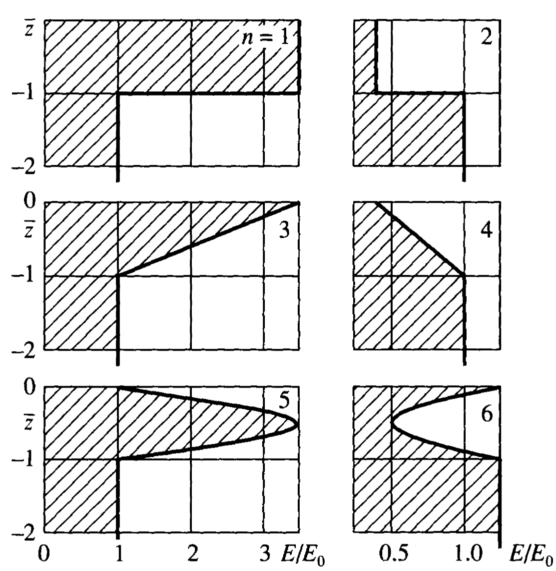

Fig. 2.

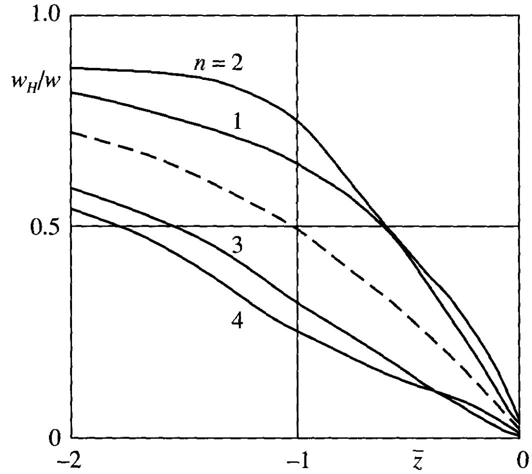

Fig. 3.

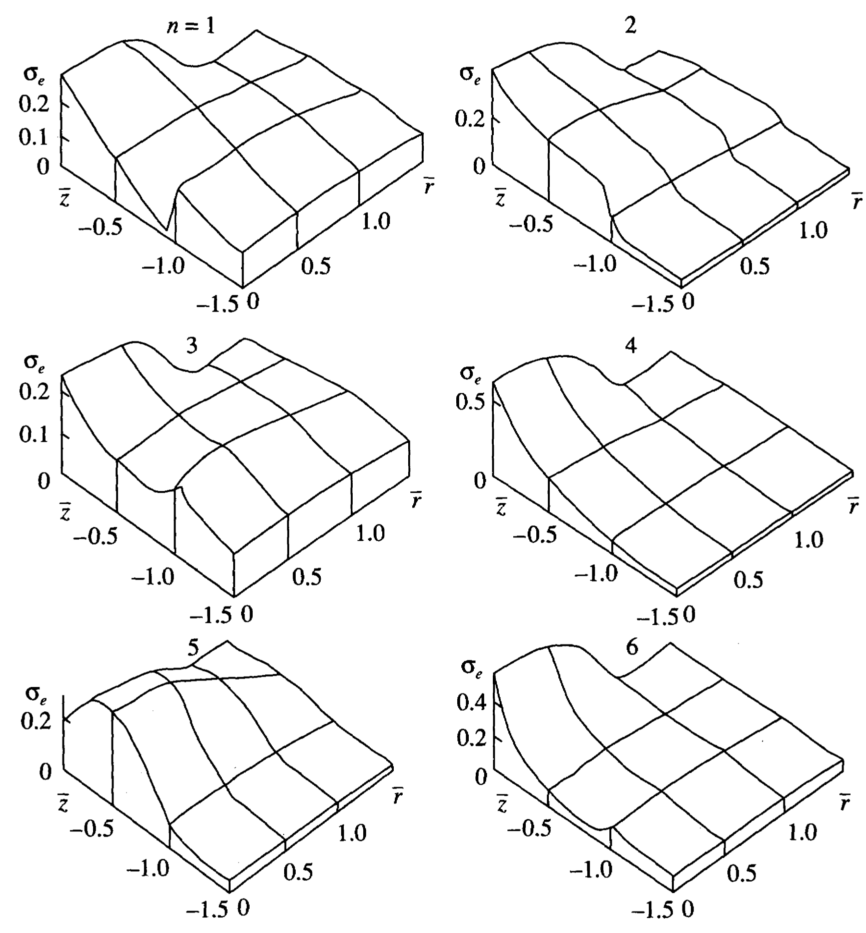

Fig. 4.

base. Note also that in the case when $n = 6$ the maximum of the intensity of the deviator stresses, as might have been expected, is inside the coating, unlike all the cases considered.

Acknowledgements

We wish to thank V. M. Aleksandrov for his interest.

This research was financed by the Russian Foundation for Basic Research (05-01-00002, 05-08-18270, 06-08-01595, 07-08-00730).

References

1. Popov GYa. The contact problem of the theory of elasticity for a circular contact area. *Prikl Mat Mekh* 1962;**26**(1):152–64.
2. Potelezhko VP, Filippov AP. The contact problem for a plate lying on an elastic base. *Prikl Mekh* 1967;**3**(1):87–91.
3. El-Sherbiney MGD, Halling J. The Hertzian contact of surfaces covered with metallic films. *Wear* 1976;**40**(3):325–37.
4. Aizikovich SM, Alexandrov VM, Kalker JJ, Krenev LI, Trubchik IS. Analytical solution of the spherical indentation problem for a half-space with gradients with the depth elastic properties. *Intern J Solids and Structures* 2002;**39**(10):2745–72.
5. Aizikovich SM. Asymptotic solutions of contact problems of the theory of elasticity for media non-uniform in depth. *Prikl Mat Mekh* 1982;**46**(1):148–58.
6. Aizikovich SM. Asymptotic solution of the problem of the interaction of a plate and a base non-uniform in depth. *Prikl Mat Mekh* 1995;**59**(4):688–97.
7. Aizikovich SM, Aleksandrov VM. The axisymmetric problem of the indentation of a circular punch into an elastic half-space non-uniform in depth. *Izv Akad Nauk SSSR MTT* 1984;**39**(2):73–82.
8. Aizikovich SM, Krenev LI, Trubchik IS. Asymptotic solution of the problem of the indentation of a spherical indentor into a half-space non-uniform in depth. *Izv Ross Akad Nauk MTT* 2000;**55**(5):107–17.
9. Aizikovich SM, Trubchik IS. Determination of the sag of the surface of a half-space non-uniform in depth when a circular punch penetrates into it. *Dokl Akad Nauk SSSR* 1993;**332**(6):702–5.
10. Aizikovich SM. Asymptotic solution of a class of paired equations. *Prikl Mat Mekh* 1990;**54**(5):872–7.
11. Gupta PK, Walowit JA. Contact stress between an elastic cylinder and a layered elastic solid. *ASME J of Lubrication Technol* 1974;**94**:250–7.
12. Leroy JM, Villechaise B. Stress determination in elastic coating and substrate under both normal and tangential loads. In: Dowson D, Taylor CM, Godet M, editors. *Mechanics of Coatings*. New York: Elsevier; 1990. p. 195–201.
13. Babeshko VA, Glushkov YeV, Glushkova NU. Methods of constructing Green's matrices for a stratified elastic half-space. *Zh Vychisl Mat Mat Fiz* 1987;**27**(1):93–101.
14. Bulychev SI, Alekhin VP. *The Testing of Materials by Continuous Indentation of an Indentor*. Moscow: Mashinostroyeniye; 1990.

Translated by R.C.G.